Department of Applied Mathematics
Preliminary Examination in Numerical Analysis
Monday August 20, 2012 (10 am - 1 pm)

Submit solutions to four (and no more) of the following six problems. Justify all your answers.

Nonlinear equations:

1. Suppose that $g : [a,b] \to [a,b]$ is continuous on the real interval $[a,b]$ and is a *contraction* in the sense that there exists a constant $\lambda \in (0,1)$ such that

$$|g(x) - g(y)| \le \lambda\, |x - y| \qquad \text{for all } x, y \in [a,b].$$

Prove that there exists a unique fixed point in $[a,b]$ and that the fixed point iteration $x_{n+1} = g(x_n)$ converges to it for any $x_0 \in [a,b]$. Also, prove that the error is reduced by a factor of at least λ *from each iteration to the next.*

Numerical quadrature:

2. We consider here three different strategies for determining weights in 3-node quadrature approximations of the form

$$\int_0^1 u(x)\,dx \approx \alpha\, u(0) + \beta\, u(\tfrac{1}{2}) + \gamma\, u(1).$$

Determine the quadrature weights (α, β, γ) that are obtained in the following three cases:

a. Trapezoidal rule,
b. Simpson's formula,
c. Exact integration of the interpolating *natural* cubic spline (i.e., the cubic spline across $[0,1]$ with end conditions that the second derivative is zero).

Interpolation / Approximation:

3. Let $f : [a,b] \to \Re$ be a real-valued continuous function on the closed interval $[a,b]$. Suppose that p_n^* solves the minimax problem in the sense that it is a polynomial of degree less than or equal to $n \ge 1$ that minimizes $\max_{x\in[a,b]} |e(x)|$ over all polynomials of degree equal to n, where $e(x) = f(x) - p_n(x)$. Prove that there must exist at least two points $\alpha, \beta \in [a,b]$, such that $|e(\alpha)| = |e(\beta)| = \max_{x\in[a,b]} |f(x) - p_n^*(x)|$ and $e(\alpha) = -e(\beta)$.