10.00AM–1.00PM, AUGUST 21, 2012

INSTRUCTIONS.

Solution sk tch s:

1· Integrating the di erential equation, we get

$$v(x) = \frac{1}{4} + \frac{1}{4}\int_0^x \sin(s + v^2(s))ds. \tag{1}$$

Let $|| \ ||_u = \sup_{\in[0,1]} |u(t)|$ denote the uniform norm, and define the set

$$X = \{\phi \in C[0,1] : \phi(0) = \frac{1}{4} \text{ and } ||\phi||_u \leq 1\}.$$

The set X combined with the uniform norm is a metric space. Now define the operator

$$[T\phi](x) = \frac{1}{4} + \frac{1}{4}\int_0^x \sin(s + (\phi(s))^2)ds;$$

the IVP can then be written as a fixed point problem $Tv = v$.

First observe that if $\phi \in X$, then $T\phi$

- Let I denote the line in the complex plane $I = \{z \in \mathbb{C} : \mathrm{Im}(z) = 0\ \mathrm{Re}(z) \in [-\frac{\pi}{2}; \frac{\pi}{2}]\}$.

(a) Set $\alpha = \beta + i\gamma$ where β and γ are real. Set $C = \sup_{, \in I} |\alpha + z| = \sqrt{(\frac{\pi}{2} + |\beta|)^2 + \gamma^2}$. Since $|[Au](x)| \leq C|u(x)|$ for all x, we get $||A|| \leq C$. For the converse, suppose that $\beta \geq 0$ (the proof for $\beta < 0$ is analogous). Set $u \ = \chi_{[\ ,\ +1]}$. Then $||u\ || = 1$ and

$$||Au\ ||^2 = \int^{+1} |(\alpha + \arctan(x))|^2\, dx = \int^{+1} \left((\beta + \arctan(x))^2 + \gamma^2 \right) dx \geq (\beta + \arctan(n))^2 + \gamma^2 \to C:$$

(b) We have

$$(Au; v) = \bar{\alpha} \int_{\mathbb{R}} \overline{u(x)}\, v(x)\, dx + \int_{\mathbb{R}} \arctan(x)\, \overline{u(x)}\, v(x)\, dx \tag{2}$$

$$(u; Av) = \alpha \int_{\mathbb{R}} \overline{u(x)}\, v(x)\, dx + \int_{\mathbb{R}} \arctan(x)\, \overline{u(x)}\, v(x)\, dx: \tag{3}$$

We see that A is self-adjoint if and only if α is real.

(c) Suppose that $Au = 0$. Then $(\alpha + \arctan(x))\, u(x) = 0$ almost everywhere. This can happen only if $u = 0$. It follows that A is one-to-one for all α.

(d) If $\alpha \notin I$, then set $\delta = \min_{, \in I} |\alpha - z| = \mathrm{dist}(I; \alpha)$. Since I is closed, $\delta > 0$. Clearly $||Au|| \geq \delta ||u||$, so A has closed range. To prove the converse, we will use that since A is one-to-one for all α, it has closed range if and only if it has a continuous inverse. Suppose first that $\alpha \in (-\pi{=}2; \pi{=}2)$. Set $I\ = (\tan(\alpha) - 1{=}n; \tan(\alpha) + 1{=}n)$ and $u\ = \chi_{I_n}$. Then $\lim_{\ \to 0} ||Au\ ||{=}||u\ || = 0$, so A does not have a bounded inverse. If $\alpha = \pm\pi$, then use $u\ = \chi_{\pm[\ ,\ +1]}$ to show that A is