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On double integrals over spheres

Radon transform. In § 4 using results of § 3 we derive some formulae in inverse scattering theory extending results of Devaney (1982a). We then discuss diffraction tomography where the inversion formula of § 3 yields a generalisation to what can be called multifrequency diffraction tomography. The same inversion formula is also used in the derivation of migration algorithms used in inverting seismic prospecting data (Beylkin and [redacted]

where ξ and η are defined as in the lemma. Using (3.10) in (3.8) and setting $r=\rho(\lambda+\mu)$ we have

$$I=\frac{\lambda\mu}{\Omega_{n-1}(\lambda+\mu)}\frac{1}{C(1-\gamma)}\int_0^\infty |p|^{n-1}\,\mathrm{d}|p|$$

$$\times\int_{S^{n-1}\times S^{n-1}} f\left(|p|\frac{(\lambda\xi+\mu\eta)}{r}\right)\frac{W(r/(\lambda+\mu))}{r\sin^{n-3}\theta}\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{3.11}$$

where

$$r=|\lambda\xi+\mu\eta|. \tag{3.12}$$

Next set

$$|p|=rk \qquad \mathrm{d}|p|=r\,\mathrm{d}k \tag{3.13}$$

to get

$$I=\frac{\lambda\mu}{2\Omega_{n-1}(\lambda+\mu)}\frac{1}{C(1-\gamma)}\int_{-\infty}^\infty |k|^{n-1}\,\mathrm{d}k$$

$$\times\int_{S^{n-1}\times S^{n-1}} f(k(\lambda\xi+\mu\eta))\frac{r^{n-1}W(r/(\lambda+\mu))}{\sin^{n-3}\theta}\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{3.14}$$

where the integral over k has been extended to the whole real axis and

$$\sin^{n-3}\theta=[1-(\xi\cdot\eta)^2]^{(n-3)/2}. \tag{3.15}$$

Inverse Fourier transform on the space $\boldsymbol{R}\times\boldsymbol{S}^{n-1}\times\boldsymbol{S}^{n-1}$

We can now formulate the following.

Theorem 1. Let g be a function on $\boldsymbol{R}^n$ and $\hat{g}$ its Fourier transform. Then

$$g(y)=\frac{1}{4(2\pi)^n\Omega_{n-1}}\int_{-\infty}^\infty |k|^{n-1}\,\mathrm{d}k\int_{S^{n-1}\times S^{n-1}}\mathrm{d}\xi\,\mathrm{d}\eta\,\frac{|\xi+\eta|^{n-1}}{[1-(\xi\cdot\eta)^2]^{(n-3)/2}}$$

$$\times W(\tfrac{1}{2}|\xi+\eta|,y)\,\hat{g}(k\xi+k\eta)\exp[\mathrm{i}(k\xi+k\eta)\cdot y] \tag{3.16}$$

where W is an arbitrary function such that

$$\int_0^1 W(\rho,y)\,\mathrm{d}\rho=1 \tag{3.17}$$

and where $\Omega_n=2\pi^{n/2}/\Gamma(\frac{1}{2}n)$ is the surface area of the unit sphere in $\boldsymbol{R}^n$.

Theorem 1 can be generalised further and leads to

Theorem 2. Let g be a function on $\boldsymbol{R}^n$ and $\hat{g}$ its Fourier transform. Let $\lambda=\lambda(y)$ and $\mu=\mu(y)$ be two positive functions on $\boldsymbol{R}^n$. Then

$$g(y)=\frac{1}{4(2\pi)^n\Omega_{n-1}}\int_{-\infty}^\infty |k|^{n-1}\,\mathrm{d}k$$

$$\times\int_{S^{n-1}\times S^{n-1}}\mathrm{d}\xi\,\mathrm{d}\eta\,b(y,\xi,\eta)\hat{g}(k\lambda\xi+k\mu\eta)\exp[\mathrm{i}(k\lambda\xi+k\mu\eta)\cdot y] \tag{3.18}$$

where

$$b(y,\xi,\eta)=\frac{|\lambda\xi+\mu\eta|^{n-1}}{\ldots}\,\frac{2\lambda\mu}{\ldots}\,\frac{1}{\ldots}\,W\left(\frac{|\lambda\xi+\mu\eta|}{\ldots},y\right) \qquad (3.19)$$

Using (2.11), (3.15), (3.19) and integrating over ψ, we have from (3.25)

$$g_{\mathrm{B}}(y)=\frac{1}{(2\pi)^n}\int_{k_{\min}}^{k_{\max}} k^{n-1}\,\mathrm{d}k\,\frac{1}{1-\gamma(y)}\int_{\gamma(y)}^{1}\mathrm{d}\rho\int_{S^{n-1}}\mathrm{d}\nu$$

$$\times W(\rho,y)\rho^n(\lambda+\mu)^n\hat{g}(k\rho(\lambda+\mu)\nu)\exp[\mathrm{i}k\rho(\lambda+\mu)\nu\cdot y]. \tag{3.26}$$

Changing the order of integration in (3.26) we arrive at

$$g_{\mathrm{B}}(y)=\frac{1}{1-\gamma(y)}\int_{\gamma(y)}^{1}\mathrm{d}\rho W(\rho,y)g_\rho(y) \tag{3.27}$$

where

$$g_\rho(y)=\frac{1}{(2\pi)^n}\int_{\rho(\lambda+\mu))k_{\min}\leqslant|p|\leqslant\rho(\lambda+\mu)k_{\max}}\mathrm{d}p\,\hat{g}(p)\exp(\mathrm{i}p\cdot y). \tag{3.28}$$

Formula (3.27) can be interpreted as a superposition of band-limited reconstructions g_ρ.

Given the definitions of the operators R^* and K we substitute (3.30) into (3.18) to obtain

$$g(y)=(R^*Kv)(y) \tag{3.34}$$

[illegible]

The validity of the Born approximation in the context of the quantum-mechanical scattering can be justified under the assumption of smallness of the potential or for high energies (frequencies) for a given potential.

Our inversion formula provides a direct reconstruction of the potential and the inter-atomic distance function for the linearised inverse problem. Indeed, applying theorem 1 specialised to dimension $n = 3$ (see also § 3, remark 1) with $g \equiv V$ and using (4.4) we have

Both of these complications (as compared with the quantum-mechanical inverse scattering problem) have been resolved. The problem with point sources was reduced more or less routinely to the problem with incident plane waves. The solution in a variable background medium can be obtained in a systematic way if we restrict ourselves to