Imaging of discontinuities in the inverse scattering problem by inversion of a causal generalized Radon transform

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This paper treats the linearized inverse scattering problem for the case of variable background velocity and for an arbitrary configuration of sources and receivers. The linearized inverse scattering problem is formulated in terms of an integral equation in a form which covers wave propagation in fluids with constant and variable densities and in elastic solids. This integral equation is connected with the causal generalized Radon transform (GRT), and an asymptotic expansion of the solution of the integral equation is obtained using an inversion procedure for the GRT. The first term of this asymptotic expansion is interpreted as a migration algorithm. As a result, this paper contains a rigorous derivation of migration as a technique for imaging discontinuities of parameters describing a medium. Also, a partial reconstruction operator is explicitly derived for a limited aperture. When specialized to a constant background velocity and specific source–receiver geometries our results are directly related to some known migration algorithms.

I. INTRODUCTION

The interpretation of seismic reflection data, ultrasound reflectivity imaging in medical applications, and various other methods of nondestructive evaluation require a solution to the inverse scattering problem. The inverse scattering problem is nonlinear and different approximate solutions have been suggested over the years. Some of the most useful in practice are the so-called *migration* schemes in geophysics. References 1–9 contain examples of such algorithms. By a migration scheme (algorithm) in this paper we understand a *technique of imaging discontinuities* of parameters describing the medium.

It must be emphasized that the construction of these approximate solutions involves (explicitly or implicitly) two major, separate steps: the first step is a linearization of the inverse problem and the second step is the solution of the linearized inverse problem.

In this paper the linearization is accomplished by a perturbation technique equivalent to the distorted wave Born approximation. We derive an integral equation formulation of the linearized inverse scattering problem for the Helmholtz equation. Analogous integral equations can be derived for fluids with variable density and for elastic solids.

The primary concern in this paper is the solution of the linearized inverse scattering problem. It requires the inversion of an integral operator with an oscillatory kernel. This operator is related—via the one-dimensional Fourier transform—to the causal generalized Radon transform (GRT).

The appearance of the GRT has a simple physical explanation. In all cases where it is impossible to make measurements directly inside the medium of interest, the only feasible measurements are *integrals* of certain combinations of parameters describing the medium. If these integrals are line integrals or integrals over hyperplanes we are dealing with the classical Radon transform.[10] Integrals with a weight function over more general hypersurfaces represent the generalized Radon transform. (Note that the problem of recovering a function from a knowledge of integrals over geometrical objects such as hypersurfaces can be viewed as a problem in the field of integral geometry.)

To solve the linearized inverse scattering problem we invert the GRT. The inversion of the GRT is of mathematical interest by itself.[11–16] In Refs. 14–16 a general explicit technique was developed for inverting the GRT. As we show here, this technique leads to an asymptotic solution of the linearized inverse problem of wave propagation.

Miller[7] recognized that seismic imaging for the general case of variable background and irregular source–receiver geometry could be cast as the problem of inverting a GRT. He derived an approximate imaging algorithm, using weighted and filtered backprojection of the data, and applied it to both synthetic and real examples. The weighting suggested in Ref. 7 differs from what we obtain by an obliquity factor.

In this paper we give an exact, formal answer to what is the proper weighting and filtering of the data, and, most important, to what is the nature of the reconstructed image.

The inversion of the GRT requires the introduction of Fourier integral operators (FIO). A special role is played by a FIO of the form $F = R^*KR$ (see Refs. 14–16). Here, R denotes the GRT, R^* is an operator dual to R, and K is a one-dimensional convolution operator; R^* is also known as the generalized backprojection operator (GBO). The Fourier integral operator F was studied in Refs. 14 and 16, and it was shown that by properly choosing the convolution operator K and the weight function of the GBO the problem of inverting the GRT can be reduced to that of solving a Fredholm integral equation.

By modifying slightly the arguments used in Refs. 14–16 and exploiting the fact that F is "almost" the identity operator we rigorously establish a class of migration algorithms as *approximate* solutions of the linearized inverse scattering problem. The approximation amounts to using only the first term of an asymptotic expansion for the "inverse" GRT. Due to the nature of the asymptotics we are able to give a precise meaning to what is reconstructed by this first-order inversion for arbitrary configurations of sources and receivers, including the case of limited view angles. In particular, we show that the (locations of) *discontinuities* of the unknown function describing the medium are recovered, rather than the function itself.

We derive an algorithm for recovering these discontinuities for variable background velocity and an arbitrary configuration of sources and receivers. Our derivation is valid as long as certain physically meaningful conditions on the global structure of rays are satisfied.

Until now, mathematically rigorous justifications relating migration to inverse scattering have been given only for migration schemes with constant background velocities and special source-receiver geometries. Previous workers, notably Norton and Linser[17] and Rose,[18] have made the connection between the Radon transform and the linearized inverse scattering problem. Norton and Linser[17] derived explicit inversion formulas for a constant background velocity and coincident sources and receivers for plane, spherical, and cylindrical apertures. They obtained certain backprojection algorithms as approximations to the exact inversion formulas. These backprojections are migration schemes in the sense of our definition. Specializing our results to their case we obtain the backprojection algorithm of Ref. 17 for a plane aperture. However, in the case of a spherical aperture our answer is different from that of Ref. 17. Our approximation remains valid even if the point of reconstruction is not close to the center of the spherical aperture. As an additional example, we obtain a migration algorithm for sources and receivers located on a plane and separated by a fixed distance.

II. LINEARIZATION OF THE INVERSE PROBLEM

To linearize the inverse scattering problem we use a standard procedure which is essentially a small perturbation technique. Formally, this procedure can be stated as follows. Consider an equation of the form

$$Lv = g, \tag{2.1}$$

where the operator L,

$$L = L_0 + L_1,$$

is a perturbation of a known operator L_0 by an operator L_1. Assuming that we can—exactly or approximately—invert the operator L_0, we look for a solution of the equation (2.1) in the form

$$v = v^{\rm in} + v^{\rm sc},$$

where

$$v^{\rm in} = L_0^{-1} g,$$

and L_0^{-1} denotes the inverse operator. Substituting this into (2.1) and applying L_0^{-1} to both sides of the equation we obtain

$$v^{\rm sc} = -L_0^{-1} L_1 v^{\rm in} - L_0^{-1} L_1 v^{\rm sc}.$$

By making the (single scattering) approximation

$$v^{\rm sc} \approx -L_0^{-1} L_1 v^{\rm in}, \tag{2.2}$$

we linearize the relation between the function $v^{\rm sc}$ and the perturbation L_1.

Let us now apply this procedure to the Helmholtz equation which describes wave propagation in a fluid of constant density. Without complicating the necessary arguments we treat this equation in n space dimensions since the dimension enters only as a parameter.

First, we consider the case where the perturbation is about a constant background velocity, which we take to be 1. Assuming then that the index of refraction $n(x)$ of the medium in some region X can be written as

$$n^2(x) = 1 + f(x),$$

the goal is to characterize the function $f(x)$ from observations of the scattered field on the boundary ∂X of the region X, as generated by a known incident field. Let us assume that the incident field is due to a point source located at a point η on the boundary ∂X. The operator L_0 is the Helmholtz operator for a constant-velocity medium, i.e.,

$$L_0 = \nabla_x^2 + k^2,$$

where ∇_x^2 is the Laplacian operator in spatial variables and the perturbation is

$$L_1 = k^2 f(x). \tag{2.3}$$

The incident field is given by the Green's function

$$v^{\rm in}(x,\eta) = -\frac{i}{4}\left(\frac{k}{2\pi|x-\eta|}\right)^{({\rm n}-2)/2} H^{(1)}_{({\rm n}-2)/2}(k|x-\eta|),$$

where $H^{(1)}_{({\rm n}-2)/2}$ is the Hankel function of the first kind. We use the first term of the asymptotic expansion of the Hankel function to approximate $v^{\rm in}$ by

$$v^{\rm in}(x,\eta) \approx e^{-i(\pi/2)({\rm n}+1)/2} \frac{k^{({\rm n}-3)/2}}{2(2\pi|x-\eta|)^{({\rm n}-1)/2}} e^{ik|x-\eta|}, \tag{2.4}$$

which is exact for n = 3. Since the operator L_0^{-1} is defined in terms of the Green's function, we obtain from (2.2) the (first term of the asymptotic expansion of) integral representation of the singly scattered field $v^{\rm sc}$ as

$$v^{\rm sc}(k,\xi,\eta) = \frac{(-ik)^{{\rm n}-1}}{4(2\pi)^{{\rm n}-1}} \int_X \frac{e^{ik|x-\xi|} e^{ik|x-\eta|}}{(|x-\xi||x-\eta|)^{({\rm n}-1)/2}} f(x)dx. \tag{2.5}$$

For n = 3, (2.5) yields $v^{\rm sc}$ as

$$v^{\rm sc}(k,\xi,\eta) = -\frac{k^2}{16\pi^2}\int_X \frac{e^{ik|x-\xi|}}{|x-\xi|}\frac{e^{ik|x-\eta|}}{|x-\eta|} f(x)dx.$$

Let us now consider the case of variable background. Assuming that the index of refraction of the medium in some region X is of the form $n^2(x) = n_0^2(x) + f(x)$, where $n_0(x)$ is known, the problem, again, is to characterize the function $f(x)$ from observation of the scattered field on the boundary ∂X. Now L_0 is the operator

$$L_0 = \nabla_x^2 + k^2 n_0^2.$$

The perturbation L_1 is of the form (2.3)

Again, we choose the incident field to be due to a point source, so that

$$(\nabla_x^2 + k^2 n_0^2) v^{\rm in}(x,\eta) = \delta(x-\eta), \tag{2.6}$$

where η indicates the position of the source. As an approximate solution of (2.6) we use, in place of (2.4), the first term of the geometrical optics approximation

$$v^{\text{in}}(x,\eta) \approx e^{-i(\pi/2)(n+1)/2} k^{(n-3)/2} A^{\text{in}}(x,\eta) e^{ik\phi^{\text{in}}(x,\eta)}. \quad (2.7)$$

Here, $\phi^{\text{in}}(x,\eta)$ is the phase function and satisfies the eikonal equation

$$(\nabla_x \phi^{\text{in}})^2 = n_0^2(x). \quad (2.8)$$

Function $A^{\text{in}}(x,\eta)$ is the amplitude and satisfies the transport equation along the ray connecting the source at η on the boundary ∂X and the point x inside the region X,

$$A^{\text{in}} \nabla_x^2 \phi^{\text{in}} + 2\nabla_x A^{\text{in}} \cdot \nabla_x \phi^{\text{in}} = 0. \quad (2.9)$$

We note that the factor $e^{-i(\pi/2)(n+1)/2} k^{(n-3)/2}$ in (2.7) is obtained by matching the geometrical optics approximation (2.7) with the exact solution in the neighborhood of the source for large k.

If we interchange source and receiver positions then (2.7) also yields the approximation $v^{\text{out}}(x,\xi)$ for the Green's function (which defines the kernel of the operator L_0^{-1}). Thus, we arrive at

$$v^{\text{out}}(x,\xi) \approx e^{-i(\pi/2)(n+1)/2} k^{(n-3)/2} A^{\text{out}}(x,\xi) e^{ik\phi^{\text{out}}(x,\xi)}, \quad (2.10)$$

where $\phi^{\text{out}}(x,\xi)$ satisfies the eikonal equation in (2.8) and $A^{\text{out}}(x,\xi)$ satisfies the transport equation along the ray connecting the point x and the receiver at the point ξ on the boundary ∂X,

$$A^{\text{out}} \nabla_x^2 \phi^{\text{out}} + 2\nabla_x A^{\text{out}} \cdot \nabla_x \phi^{\text{out}} = 0. \quad (2.11)$$

In general, one can solve the eikonal equation (2.8) by ray tracing. The transport equations in (2.9) and (2.11) then reduce to ordinary differential equations along rays.[19] If the background index of refraction $n_0(x)$ is discontinuous then the rays satisfy Snell's law on surfaces of discontinuities and appropriate transmission coefficients have to be used in computing the amplitudes on these surfaces. We formulate the assumptions we need to make about the global structure of rays—and, therefore, about the background index of refraction—in the next section.

Combining (2.2), (2.3), (2.7), and (2.10) we find the (first term of the asymtotic expansion of) integral representation of the singly scattered field

$$v^{\text{sc}}(k,\xi,\eta) = (-ik)^{n-1} \int_X e^{ik\phi^{\text{out}}(x,\xi)} e^{ik\phi^{\text{in}}(x,\eta)} \times A^{\text{out}}(x,\xi) A^{\text{in}}(x,\eta) f(x) dx, \quad (2.12)$$

as a function of the receiver position ξ, the source position η, and wave number k. The integral representation (2.12) is an integral equation for the unknown function f.

Analogous integral equations can be derived for fluids with variable density and for elastic solids. We shall present the derivation elsewhere. In these cases integral equations of the type in (2.12) relate the singly scattered field to a combination of parameters characterizing the medium. For elastic media we obtain a system of four integral equations corresponding to p–p, p–s, s–p, and s–s scattered fields, and the phase functions ϕ^{in} and ϕ^{out} satisfy different eikonal equations corresponding to the indices of refraction of p and s waves. The amplitudes $A^{\text{in}}(x,\eta)$ and $A^{\text{out}}(x,\xi)$ satisfy the corresponding transport equations along the rays connecting points x,η and x,ξ, respectively. In the following sections we consider the integral equation (2.12) in a form which covers these cases.

III. THE INTEGRAL EQUATION OF THE LINEARIZED INVERSE PROBLEM AND THE CAUSAL GRT

It was shown in the previous section that the linearization of the inverse scattering problem leads to the integral equation

$$v(k,\xi,\eta) = (-ik)^{n-1} \int_X f(x) a(x,\xi,\eta) e^{ik\phi(x,\xi,\eta)}\, dx, \quad (3.1)$$

where X is the domain of definition of the unknown function $f(x)$, ξ and η are points on the boundary ∂X corresponding to receiver and source locations, and k is the wave number. The phase function $\phi(x,\xi,\eta)$ is the sum of two phase functions

$$\phi(x,\xi,\eta) = \hat{\phi}(x,\xi) + \tilde{\phi}(x,\eta), \quad (3.2)$$

which satisfy the eikonal equations

$$(\nabla_x \hat{\phi}(x,\xi))^2 = \hat{n}^2(x), \quad (3.3a)$$

and

$$(\nabla_x \tilde{\phi}(x,\eta))^2 = \tilde{n}^2(x). \quad (3.3b)$$

In (3.3a) and (3.3b), $\hat{n}$ and $\tilde{n}$ are indices of refraction, i.e., positive bounded functions. We have replaced ϕ^{out} and ϕ^{in} by $\hat{\phi}$ and $\tilde{\phi}$. The function $a(x,\xi,\eta)$ in (3.1) replaces the product of the amplitudes A^{out} and A^{in}

$$a(x,\xi,\eta) = A^{\text{out}}(x,\xi) A^{\text{in}}(x,\eta). \quad (3.4)$$

Both A^{out} and A^{in} are positive since they are solutions of the transport equations in (2.9) and (2.11). This is true even for discontinuous indices of refraction $\hat{n}$ and $\tilde{n}$ as long as the global structure of rays satisfies the assumptions formulated later in this section. Therefore, $a(x,\xi,\eta)$ is positive on $X \times \partial X \times \partial X$ and can be called a weight function. We assume, in addition, that $a(x,\xi,\eta)$ is infinitely differentiable, namely, $a(x,\xi,\eta) \in C^\infty(\overline{X} \times \partial X \times \partial X)$, where $\overline{X}$ is any compact set contained in X.

The integral equation (3.1) is related to a causal GRT. To see this, consider the transform R defined by

$$(Rf)(t,\xi,\eta) = \int f(x) a(x,\xi,\eta) \delta(t - \phi(x,\xi,\eta)) dx, \quad \text{for } t \geqslant 0,$$
$$(Rf)(t,\xi,\eta) = 0, \quad \text{for } t \leqslant 0. \quad (3.5)$$

We call the transform R in (3.5) the causal GRT for obvious reasons and note that the transform R agrees with the GRT as defined in Refs. 14–16 for $t \geqslant 0$. However, the integral in (3.5) is not defined for $t<0$, and it is natural to set $(Rf)(t,\xi,\eta) = 0$ for $t \leqslant 0$. Since in this article we consider only the causal GRT, we drop the word causal.

The Fourier transform $(Rf)\hat{}(k,\xi,\eta)$ of $(Rf)(t,\xi,\eta)$ in (3.5) with respect to t yields the integral in (3.1) up to the factor $(-ik)^{n-1}$, since the function $v(k,\xi,\eta)/(-ik)^{n-1}$ can be shown to satisfy the dispersion relation if $\hat{\phi}$ and $\tilde{\phi}$ are positive. Thus,

$$v(k,\xi,\eta) = (-ik)^{n-1} (Rf)\hat{}(k,\xi,\eta). \quad (3.6)$$

We consider the problem of finding the function $f(x)$ in (3.1) in the following two situations: (i) the position of the source is fixed, i.e., we are given $v(k,\xi,\eta)$ for fixed η and for a set of values $\xi \in \partial X$; and (ii) the position of the receiver is a function of the position of the source, i.e., we are given $v(k,\xi(\eta),\eta)$, where $\xi(\eta)$ is a known function of η, for a set of values $\eta \in \partial X$.

We specialize the arguments of Refs. 14 and 16 to the case of the integral equation in (3.1). Having established the relation of the integral equation in (3.1) to the GRT it becomes natural to introduce the same generalized backprojection operator and Fourier integral operator used in Refs. 14–16 to study the integral equation in (3.1). This we do in Sec. IV of this article.

We make the following assumptions about the domain X, its boundary ∂X, and indices of refraction in (3.3a) and (3.3b).

Let $n(x)$ be the index of refraction in the region X and let S_x^{n-1} be the unit sphere with the center at the origin of the tangent space at the interior point $x \in X$. Here, S_x^{n-1} represents all directions at the point $x \in X$. Let $\{\gamma(x,\xi)\}$ be a family of geodesics (rays) of the metric $n(x)dx$ connecting the point x with points $\xi \in \partial X^0$, where $\partial X^0 \subset \partial X$ is an open region of the boundary. Each ray within the family has a direction $\omega \in S_x^{n-1}$ at the point x. Thus the family of rays maps directions at the point x (an open domain of the unit sphere S_x^{n-1}) into ∂X^0. *In this article we always assume that this map is an orientation-preserving diffeomorphism.* Algebraically this means that certain Jacobians do not vanish. Physically it means that if a source located at an interior point of X illuminates a region ∂X^0 on the boundary, then this region can be smoothly contracted along the rays into a part of a small sphere around the source. We note that this assumption leads to the uniqueness and stability estimate in the inverse travel time problem.[20] When the index of refraction is constant this condition is satisfied for domains which are star shaped with respect to points of reconstruction. These include all practical configurations in geophysics, tomography, and nondestructive testing.

Our next remark deals with the domains of definition of the operators that appear in this article. We always define operators on functions which belong to the class $C_0^\infty(X)$ or $C^\infty(X)$. However, we can extend the domain of definition to the appropriate dual class of generalized functions by the standard procedure (see Appendix B). Thereby, we consider such an extension automatically performed each time we define an operator in this paper.

IV. ASYMPTOTIC SOLUTION OF THE LINEARIZED INVERSE PROBLEM WITH A FIXED INCIDENT FIELD

In this section we construct an asymptotic solution of the integral equation in (3.1) given the function $v(k,\xi,\eta)$, where η—the position of the source—is fixed. For the sake of brevity the dependence on η will sometimes be suppressed. Thus, we write the integral equation in (3.1) as

$$v(k,\xi) = (Wf)(k,\xi), \tag{4.1}$$

where

$$(Wf)(k,\xi) = (-ik)^{n-1}\int_X f(x)a(x,\xi)e^{ik\phi(x,\xi,\eta)}dx. \tag{4.2}$$

In (4.1) and (4.2), $v(k,\xi)$ and $a(x,\xi)$ stand for $v(k,\xi,\eta)$ and $a(x,\xi,\eta)$ in (3.1). The phase function $\phi(x,\xi,\eta)$ is described in (3.2).

We now introduce the generalized backprojection operator R^* dual to the generalized Radon transform R. For infinitely differentiable functions $u(t,\xi) \in C^\infty(R_+ \times \partial X)$ we define R^* as

$$(R^*u)(y) = \int_{\partial X} u(t,\xi)\Big|_{t=\phi(y,\xi,\eta)} b(y,\xi)d\xi. \tag{4.3}$$

The weight function $b(y,\xi)$ is a smooth, non-negative function on $X \times \partial X$, $b(y,\xi) \in C^\infty(X \times \partial X)$, which we have chosen to be

$$b(y,\xi) = [h(y,\xi)/a(y,\xi)]\,\chi(y,\xi), \tag{4.4}$$

where $h(y,\xi)$ is the determinant

$$h(y,\xi) = \begin{bmatrix} \phi_{y_1} & \phi_{y_2} & \cdots & \phi_{y_n} \\ \hat{\phi}_{y_1\xi_1} & \hat{\phi}_{y_2\xi_1} & \cdots & \hat{\phi}_{y_n\xi_1} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\phi}_{y_1\xi_{n-1}} & \hat{\phi}_{y_2\xi_{n-1}} & \cdots & \hat{\phi}_{y_n\xi_{n-1}} \end{bmatrix}, \tag{4.5}$$

and $\chi(y,\xi)$ is a cutoff function. The cutoff function $\chi(y,\xi)$ is described below and is chosen to ensure that $\chi(y,\xi)h(y,\xi) \geqslant 0$ on $X \times \partial X$. The phase functions ϕ, $\hat{\phi}$, and $\tilde{\phi}$ are related by (3.2). The functions $\hat{\phi}$ and $\tilde{\phi}$ are solutions of the eikonal equations (3.3a) and (3.3b). Our particular choice (4.4) of the weight function $b(y,\xi)$ in (4.3) makes it possible to regularize the problem of inverting the GRT as was shown in Refs. 14 and 16.

We now change the variable of integration in (4.3). For each point y in the interior of X let ω denote a point on the unit sphere S_y^{n-1}. This means that ω is a direction at the point y. For the ray with direction ω at the interior point $y \in X$ let $\xi(\omega)$ be the point of intersection of that ray with the boundary ∂X. According to the assumptions formulated in the previous section, the function $\xi = \xi(\omega)$ is invertible and has continuous partial derivatives of the first order. We choose $\omega \in S_y^{n-1}$ to be the new variable of integration in (4.3). From Lemma A in Appendix A it follows that

$$h(y,\xi)d\xi = \hat{n}^n(1 + (\tilde{n}/\hat{n})\cos\psi)d\omega, \tag{4.6}$$

where

$$\cos\psi(y,\xi) = [\nabla_y\hat{\phi}(y,\xi)\cdot\nabla_y\tilde{\phi}(y,\eta)]/\hat{n}(y)\tilde{n}(y), \tag{4.7}$$

and $d\omega$ is the standard solid angle differential form on the unit sphere S_y^{n-1}.

Substituting (4.6) in (4.3) we rewrite R^* as

$$(R^*u)(y) = \hat{n}^n(y)\int_{S_y^{n-1}} u(t,\xi)\Big|_{t=\phi(y,\xi,\eta)} \frac{\chi(y,\xi)}{a(y,\xi)} \times\left[1 + \frac{\tilde{n}(y)}{\hat{n}(y)}\cos\psi(y,\xi)\right]d\omega, \tag{4.8}$$

where $\xi = \xi(\omega)$. In this form the operator R^* can be computed explicitly by making use of ray tracing.

It remains to define the cutoff function $\chi(y,\xi)$. Given the interior point $y \in X$ and the boundary point $\xi \in \partial X$ we set $\chi(y,\xi) = 0$, if $1 + [\tilde{n}(y)/\hat{n}(y)]\cos\psi(y,\xi) \leqslant 0$. Choosing an arbitrarily small $\epsilon > 0$ we set $\chi(y,\xi) = 1$, if $1 + [\tilde{n}(y)/\hat{n}(y)]\cos\psi(y,\xi) \geqslant \epsilon$, and define $\chi(y,\xi)$ elsewhere so that it is infinitely differentiable and $0 \leqslant \chi(y,\xi) \leqslant 1$.

Let $\partial X_\eta(y)$ be the region of the boundary defined by

$$\partial X_\eta(y) = \{\xi \in \partial X: \cos\psi(y,\xi) < -[\hat{n}(y)/\tilde{n}(y)] + \epsilon\}.$$

Let $\partial X_\eta^0(y)$ be the complement of $\partial X_\eta(y)$, i.e.,

$$\partial X_\eta^0(y) = \partial X \setminus \partial X_\eta(y).$$

To describe $\partial X_\eta(y)$ and, therefore, $\partial X_\eta^0(y)$ for a given interior point $y \in X$, three cases can be distinguished.

(1) $\hat{n}(y) > \tilde{n}(y)$. In this case $\partial X_\eta(y)$ is empty since ϵ can be chosen sufficiently small.

(2) $\hat{n}(y) = \tilde{n}(y)$. In this case for sufficiently small ϵ the region $\partial X_\eta(y)$ consists of an ϵ-neighborhood of the unique point ξ_0, where $\cos \psi(y,\xi_0) = -1$.

(3) $\hat{n}(y) < \tilde{n}(y)$. In this case for any sufficiently small ϵ the region $\partial X_\eta(y)$ is a connected part of the boundary ∂X.

The cutoff function $\chi(y,\xi) \equiv 1$ on $\partial X_\eta^0(y)$. If (1) holds the cutoff function $\chi(y,\xi) \equiv 1$ on all of ∂X. If (2) holds the cutoff function isolates a single point ξ_0 and is introduced here for technical reasons. In carrying out the integration in (4.8) one can set the cutoff function $\chi(y,\xi) \equiv 1$ on the boundary ∂X. If (3) holds the cutoff function $\chi(y,\xi)$ is zero on all but a small open subset of $\partial X_\eta(y)$ which is determined by the choice of ϵ. In carrying out the integration in (4.8) we can set $\epsilon = 0$ and $\chi(y,\xi) = 0$ on $\partial X_\eta(y)$.

If $v(k,\xi)$ is available only on a part of the boundary ∂X we have to modify the cutoff function. We include an ϵ-neighborhood of the region where the function $v(k,\xi)$ is not known in the set $\partial X_\eta(y)$. Then $\chi(y,\xi)$ is again defined in such a way that it is infinitely differentiable with values $0 \leqslant \chi(y,\xi) \leqslant 1$ with $\chi(y,\xi) \equiv 1$ on $\partial X_\eta^0(y) = \partial X \setminus \partial X_\eta(y)$ and $\chi(y,\xi) = 0$ on all but an arbitrary small subset within $\partial X_\eta(y)$. We denote the modified cutoff function again by $\chi(y,\xi)$.

Note, that if (3) holds—as in the case when the incoming wave is *s–p* converted at the point $y \in X$ in an elastic medium—the angle ψ_0, such that $\cos \psi_0 = -\hat{n}(y)/\tilde{n}(y)$ plays the role of a "critical angle" between the directions of incoming and outgoing waves at the point y.

We now consider the FIO defined by

$$(Ff)(y) = \frac{1}{(2\pi)^n} \int_0^\infty \int_{\partial X} \int_X e^{ik\Phi(x,y,\xi)} A(x,y,\xi) \times f(x) dx\, d\xi\, k^{n-1}\, dk, \tag{4.9}$$

where

$$\Phi(x,y,\xi) = \phi(x,\xi,\eta) - \phi(y,\xi,\eta). \tag{4.10}$$

The function $\phi(x,\xi,\eta)$ is described in (3.2), and

$$A(x,y,\xi) = [a(x,\xi)/a(y,\xi)] h(y,\xi) \chi(y,\xi), \tag{4.11}$$

where $a(x,\xi)$ is the weight function in (4.2). Let us also introduce the operator $\mathscr{F}^+$:

$$(\mathscr{F}^+ v)(t) = \frac{i^{n-1}}{(2\pi)^n} \int_0^\infty v(k) e^{-ikt}\, dk. \tag{4.12}$$

We have

$$F = R^* \mathscr{F}^+ W. \tag{4.13}$$

To investigate the operator F we observe that the first term in the Taylor series for $\Phi(x,y,\xi)$ is $\nabla_y \phi(y,\xi,\eta) \cdot (x-y)$ and consider the operator

$$(I_{\partial X_\eta^0} f)(y) = \frac{1}{(2\pi)^n} \int_0^\infty \int_{\partial X_\eta^0(y)} \int_X e^{ik\nabla_y \phi(y,\xi,\eta)\cdot(x-y)} \times h(y,\xi) f(x) dx\, d\xi\, k^{n-1}\, dk. \tag{4.14}$$

Changing variables of integration from k,ξ to p, where

$$p = k \nabla_y \phi(y,\xi,\eta), \tag{4.15}$$

we find that

$$dp = k^{n-1} h(y,\xi) d\xi\, dk, \tag{4.16}$$

and, thereby,

$$(I_{\partial X_\eta^0} f)(y) = \frac{1}{(2\pi)^n} \int_{\Omega_\eta(y)} \int_X e^{ip\cdot(x-y)} f(x) dx\, dp, \tag{4.17}$$

where $\Omega_\eta(y)$ is the image of $R_+ \times \partial X_\eta^0(y)$ under the change of variables in (4.15). It follows from (4.17) that

$$(I_{\partial X_\eta^0} f)(y) = \frac{1}{(2\pi)^n} \int_{\Omega_\eta(y)} e^{-ip\cdot y} f\hat{\ }(p) dp, \tag{4.18}$$

where $f\hat{\ }(p)$ is the Fourier transform of the function f. It is clear that if $\partial X_\eta^0(y) = \partial X$ then the operator $I_{\partial X_\eta^0}$ is the identity operator. If $\partial X_\eta^0(y) \neq \partial X$, then the operator $I_{\partial X_\eta^0}$ in (4.18) is *an operator of partial reconstruction*.

It is important to note that for each point $y \in X$, the region $\partial X_\eta^0(y)$ on the boundary ∂X can be explicitly constructed by ray tracing. Having found $\partial X_\eta^0(y)$, we can then determine explicitly the set $\Omega_\eta(y)$ in the Fourier domain, where the Fourier transform $f\hat{\ }(p)$ of the function f, which we would like to recover, is known. If only partial data are available this set $\Omega_\eta(y)$ in the Fourier domain determines the spatial resolution of the partial reconstruction (4.18) and controls what can be recovered in the migration schemes presented below.

The asymptotic solution of the integral equation in (4.1) is constructed by making use of the following theorem.

Theorem 1: The Fourier integral operator F in (4.9) is a pseudodifferential operator and can be represented as a sum

$$F = I_{\partial X_\eta^0} + T_1 + T_2 + \cdots, \tag{4.19}$$

where $I_{\partial X_\eta^0}$ denotes the operator described in (4.17) and the operators $T_1, T_2, \ldots$ belong to increasingly smooth classes of pseudodifferential operators.

The definition of classes of pseudodifferential operators can be found in Appendix B. For further references see Ref. 21, or any other reference where Fourier integral operators and pseudodifferential operators are studied.

It follows from (4.13) and Theorem 1 that by making use only of the first term in (4.19),

$$R^* \mathscr{F}^+ W \approx I_{\partial X_\eta^0}, \tag{4.20}$$

we obtain an approximate reconstruction algorithm. The expansion in (4.19) also explains the precise meaning of the approximation in (4.20). Since we neglect all terms in the expansion which appear to be smoothing operators, the approximation in (4.20) reconstructs only (the location of) the discontinuities of the function f (or the places, where the gradient of f is large). In this sense the formula in (4.20) provides an algorithm for imaging the discontinuities. This is, of course, what is sought in geophysics and many other applications where the discontinuities of parameters describing the medium are of interest. In Sec. VI we describe the algorithm contained in (4.20) in greater detail.

The remainder of this section contains an outline of the proof of Theorem 1. The material presented in Secs. V and VI is independent of the details of the proof.

The proof follows along the same lines as the arguments presented in Refs. 14 and 16. Consider the set

$$C_\Phi = \{(k,\xi,x,y) \in R_+ \times \partial X_\eta^0(y) \times X \times X : \Phi(x,y,\xi) = 0, \quad \nabla_\xi \Phi(x,y,\xi) = 0\}. \tag{4.21}$$

This set is of fundamental importance in the theory of Fourier integral operators since its structure determines the properties of the operator. The definition (4.21) is not standard (see Ref. 21, for example), however the change of variables (4.15) transforms it into the classical one.

Using the assumption that the function $\xi = \xi(\omega)$ is a diffeomorphism it can be shown that

$$C_\Phi = \{(k,\xi,x,x): k \in R_+,\ \xi \in \partial X_\eta^0(y),\ x \in X\}, \tag{4.22}$$

so that the projection of C_Φ on $X \times X$ is the diagonal. This implies that the operator in (4.9) is a pseudodifferential operator as defined in Appendix B.

Let us consider $\chi_\delta(x,y) \in C^\infty(X \times X)$, $0 \leqslant \chi_\delta(x,y) \leqslant 1$, such that

$$\chi_\delta(x,y) = 1, \text{ if } |x-y| < \delta/2,$$
$$\chi_\delta(x,y) = 0, \text{ if } |x-y| > \delta,$$

where $\delta > 0$ is an arbitrary small parameter. Instead of the FIO (4.9) we can study the following operator (we keep the same notation):

$$(Ff)(y) = \frac{1}{(2\pi)^n} \int_0^\infty \int_{\partial X} \int_X e^{ik\Phi(x,y,\xi)} A(x,y,\xi) \times \chi_\delta(x,y) f(x) dx\, d\xi\, k^{n-1}\, dk. \tag{4.23}$$

This operator differs from the operator in (4.9) by a regularizing operator (see Appendix B). The regularizing operator does not change the asymptotics and can be neglected since it is "infinitely smooth."

If δ is sufficiently small and $|x-y| < \delta$ we can write the phase function $\Phi(x,y,\xi)$ as

$$\Phi(x,y,\xi) = \nabla_y \phi(y,\xi,\eta) \cdot (x-y) + H(x,y,\xi), \tag{4.24}$$

where $H(x,y,\xi) = O(|x-y|^2)$, and the amplitude

$$A(x,y,\xi) = h(y,\xi)\chi(y,\xi) + \tilde{A}(x,y,\xi), \tag{4.25}$$

where $\tilde{A}(x,y,\xi) = O(|x-y|)$. Making the change of variables (4.15) and using (4.16), (4.23) becomes

$$(Ff)(y) = \frac{1}{(2\pi)^n} \int_{\Omega_\eta(y)} \int_X e^{ip\cdot(x-y) + iH(x,y,p)} \times (1 + \tilde{A}(x,y,p)) f(x) dx\, dp, \tag{4.26}$$

where $H(x,y,p) = k(p)H(x,y,\xi(p))$ and $\tilde{A}(x,y,p) = \tilde{A}(x,y,\xi(p))$. The functions $k(p)$ and $\xi(p)$ can be determined from the change of variables in (4.15). Note, that as functions of p, $H(x,y,p)$ and $\tilde{A}(x,y,p)$ are homogeneous of degree 1 and 0, respectively. Consider now the operator

$$(F_s f)(y) = \frac{1}{(2\pi)^n} \int_{\Omega_\eta(y)} \int_X e^{ip\cdot(x-y) + isH(x,y,p)} \times (1 + s\tilde{A}(x,y,p)) f(x) dx\, dp. \tag{4.27}$$

Since $F = F_1$ we can use the Taylor expansion of $F_s f$ as a function of s to express F as

$$F = \sum_{m=0}^{N} \frac{1}{m!} \left(\frac{d}{ds}\right)^m F_s \bigg|_{s=0} + \int_0^1 \frac{(1-s)^N}{N!} \left(\frac{d}{ds}\right)^{N+1} F_s\, ds. \tag{4.28}$$

It was shown[14,16] that the expansion in (4.28) of the operator F of the form in (4.27) consists of increasingly smooth pseudodifferential operators. Comparing the first term in the expansion (4.28) with (4.17) we obtain the expansion in (4.19). One can compute and use more terms in the expansion (4.19). For example, the operator T_1 was computed in Refs. 14–16.

V. ASYMPTOTIC SOLUTION OF THE LINEARIZED INVERSE PROBLEM WHEN THE RECEIVER POSITIONS DEPEND ON THE SOURCE POSITIONS

In this section we construct an asymptotic solution of the integral equation in (3.1) given the function $v(k,\xi,\eta)$, where the receiver positions $\xi = \xi(\eta)$ depend on the source positions η. The arguments in this case are analogous to those for the fixed source position, and are presented briefly for this reason. Again, consider the integral equation in (3.1), which we now write as

$$v(k,\eta) = (Wf)(k,\xi(\eta),\eta), \tag{5.1}$$

where

$$(Wf)(k,\xi(\eta),\eta) = (-ik)^{n-1} \int_X f(x) a(x,\xi(\eta),\eta) \times e^{ik\phi(x,\xi(\eta),\eta)}\, dx. \tag{5.2}$$

The phase function ϕ is described in (3.2).

For functions $u(t,\eta) \in C^\infty(R_+ \times \partial X)$ we define the dual transform R^* as

$$(R^*u)(y) = \int_{\partial X} u(t,\eta) \Big|_{t=\phi(y,\xi(\eta),\eta)} b(y,\eta) d\eta, \tag{5.3}$$

where the weight function $b(y,\eta)$ is a smooth, non-negative function on $X \times \partial X$, $b(y,\eta) \in C^\infty(X \times \partial X)$ and is chosen to be

$$b(y,\eta) = [h(y,\eta)/a(y,\xi(\eta),\eta)]\chi(y,\eta). \tag{5.4}$$

Here, $h(y,\eta)$ is the determinant

$$h(y,\eta) = \begin{bmatrix} \phi_{y_1} & \phi_{y_2} & \cdots & \phi_{y_n} \\ \phi_{y_1\eta_1} & \phi_{y_2\eta_1} & \cdots & \phi_{y_n\eta_1} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{y_1\eta_{n-1}} & \phi_{y_2\eta_{n-1}} & \cdots & \phi_{y_n\eta_{n-1}} \end{bmatrix}, \tag{5.5}$$

and $\chi(y,\eta)$ is a cutoff function described below. Note, that the function $h(y,\eta)$ differs from the one in (4.5) in the previous section. To compute the determinant (5.5) we again use Lemma A in Appendix A. In the two-dimensional case we find that

$$h(y,\eta) d\eta = \left(\hat{n}^2 \frac{d\xi}{d\eta} + \tilde{n}^2 + \hat{n}\tilde{n}\left(1 + \frac{d\xi}{d\eta}\right)\cos\psi\right) d\omega, \tag{5.6}$$

where

$$\cos\psi(y,\eta) = [\nabla_y \hat{\phi}(y,\xi(\eta)) \cdot \nabla_y \tilde{\phi}(y,\eta)]/\hat{n}(y)\tilde{n}(y), \tag{5.7}$$

and $d\omega$ is the standard angle measure on the unit circle.

For simplicity let us assume that the function $\xi(\eta)$ is such that the function $h(y,\eta)$ in (5.5) is strictly positive for all $y \in X$ and $\eta \in \partial X$. In this case the infinitely differentiable cutoff function—$\chi(y,\eta)$—is introduced only to isolate a region of the boundary $\partial\tilde{X}(y)$, where we do not know the function $v(k,\eta)$. The cutoff function $\chi(y,\eta) = 0$ for points η in $\partial\tilde{X}(y)$; it has values $0 \leqslant \chi(y,\xi) \leqslant 1$ in an ϵ-neighborhood of $\partial\tilde{X}(y)$, where ϵ is arbitrarily small, and it is set equal to 1 elsewhere. Let $\partial X^0(y)$ denote the complement of $\partial\tilde{X}(y)$, i.e.,

$$\partial X^0(y) = \partial X \setminus \partial\tilde{X}(y).$$

Again, consider the Fourier integral operator F

$$(Ff)(y) = \frac{1}{(2\pi)^n}\int_0^\infty \int_{\partial X}\int_X e^{ik\Phi(x,y,\eta)} \times A(x,y,\eta) f(x) dx\, d\eta\, k^{n-1}\, dk, \quad (5.8)$$

where Φ is defined in terms of the function ϕ in (3.2) as

$$\Phi(x,y,\eta) = \phi(x,\xi(\eta),\eta) - \phi(y,\xi(\eta),\eta), \quad (5.9)$$

and

$$A(x,y,\eta) = \frac{a(x,\xi(\eta),\eta)}{a(y,\xi(\eta),\eta)} h(y,\eta)\chi(y,\eta). \quad (5.10)$$

Using the operator $\mathscr{F}^+$ defined in (4.12), we have

$$F = R^*\mathscr{F}^+ W. \quad (5.11)$$

The analysis of the operator F is conducted analogously to the one in the previous section. In this case the change of the variables of integration in (5.8) from k,η to p is as follows:

$$p = k\nabla_y\phi(y,\xi(\eta),\eta), \quad (5.12)$$

and

$$dp = k^{n-1}h(y,\eta)d\eta\, dk. \quad (5.13)$$

The asymptotic solution of the integral equation in (5.1) is constructed by making use of the following theorem.

Theorem 2: The Fourier integral operator in (5.8) is a pseudodifferential operator and can be represented as a sum

$$F = I_{\partial X^0} + T_1 + T_2 + \cdots, \quad (5.14)$$

where $I_{\partial X^0}$ denotes the operator

$$(I_{\partial X^0}f)(y) = \frac{1}{(2\pi)^n}\int_{\Omega(y)} e^{-ip\cdot y}\hat{f}(p)dp, \quad (5.15)$$

where $\Omega(y)$ is the image of $R_+\times\partial X^0(y)$ under the change of variables in (5.12). The operators $T_1,T_2,\ldots$ belong to increasingly smooth class of pseudodifferential operators (see Appendix B).

The first term of the expansion in (5.14) yields the approximate reconstruction algorithm

$$R^*\mathscr{F}^+W \approx I_{\partial X^0}, \quad (5.16)$$

where the generalized backprojection operator R^* is given by (5.3).

In the following section we show that for constant background and coincident sources and receivers the approximation (5.16) reduces to algorithms described in the literature.

VI. THE ASYMPTOTIC SOLUTIONS AND MIGRATION SCHEMES

This section contains a brief description of migration schemes which follow from our results. As we shall see, the measured scattered data are such that the migration schemes amount to the generalized backprojections (except when the Hilbert transform has to be applied first in spaces of even dimensions).

Let us recall that the goal is to estimate the unknown function $f(x)$ in (2.12) or (3.1) from observations of the (singly) scattered field. We assume that the scattered field $u = u^{sc}$ is given in the time domain, so that

$$u(t,\xi,\eta) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} v(k,\xi,\eta)e^{-ikt}\, dk, \quad (6.1)$$

where $v(k,\xi,\eta)$ is described in (3.1). In many cases of practical interest the actual measurements are measurements of the total field, and the incident field must, by some means, be removed. However, in this paper we assume that the (singly) scattered field is given in the time domain to start with.

Comparing the definition of the operator $\mathscr{F}^+$ in (4.12) with the Fourier transform in (6.1) and denoting the real part of the operator (4.12) as $w(t,\xi,\eta) \equiv \mathrm{Re}(\mathscr{F}^+v)(t,\xi,\eta)$ we obtain

$$w(t,\xi,\eta) = [(-1)^{(n-1)/2}/2(2\pi)^{n-1}]u(t,\xi,\eta), \quad (6.2)$$

in spaces of odd dimensions $n = 2m+1$, $m = 1,2,\ldots$, and

$$w(t,\xi,\eta) = \frac{(-1)^{n/2}}{2(2\pi)^{n-1}}\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{u(t',\xi,\eta)}{t-t'}dt', \quad (6.3)$$

in spaces of even dimensions $n = 2m$, $m = 1,2,\ldots$. Thus, one has to apply the Hilbert transform (6.3) to the scattered field in spaces of even dimensions to obtain $\mathrm{Re}(\mathscr{F}^+v)(t,\xi,\eta)$.

It follows from (6.2) and (6.3) that the only remaining step in algorithms (4.20) and (5.16) is to construct the generalized backprojection operator (GBO) in (4.3) or (5.3), depending on the source-receiver configuration. Let us consider the case when the receiver position ξ depends on the source position η. The construction of the GBO for fixed source position in (4.3) is completely analogous.

To compute the GBO (5.3) we have to compute both the phase function and the weight function. Such a computation is equivalent to the construction of two Green's functions in ray approximation. Indeed, the computation of the functions $\phi^{\text{in}} = \tilde{\phi}$ and $\phi^{\text{out}} = \hat{\phi}$ in (3.2) and the factors A^{in} and A^{out} in (3.4)—which are necessary to construct the weight function in (5.3) [or (4.3)]—amounts to the computation of the ray approximation of two Green's functions along the two rays connecting the point of interest in the medium with the source and with the receiver. The additional obliquity factor in the weight function can be easily computed as it follows from (5.6) and (5.7) [or (4.6) and (4.7)]. This factor depends on the angle between the rays connecting the point of interest in the medium with the source and receiver.

Once both the phase function and the weight function are computed the GBO is applied in the time domain, either to the singly scattered field itself [in spaces of odd dimensions (6.2)] or to the Hilbert transform of the singly scattered field [in spaces of even dimensions (6.3)], as required by the approximate formulas in (4.20) or (5.16). In this way we obtain the reconstruction f_{mig} of the function f from (6.2), (6.3) and (5.16) as

$$f_{\text{mig}} \approx \mathrm{Re}\, I_{\partial X^0} f. \quad (6.4)$$

If $\partial X^0 = \partial X$ then the operator $I_{\partial X^0}$ is the identity operator. The function f in (2.12) [or (3.1)] is assumed to be real, and, therefore, it follows from (6.4) that $f_{\text{mig}}(x) \approx f(x)$ in the region X. The symbol $\approx$ expresses the fact that we *image the* (*location of*) *discontinuities* of the function $f(x)$ in the region X, since smooth terms in the asymptotic expansion in (5.14) are neglected in the approximation.

In most practical situations we have data only for limited view angles, and, therefore, $\partial X^0 \neq \partial X$. Thus, we obtain a partial reconstruction since we can recover the Fourier transform $\hat{f}$ [see (5.15)] only on a part of the Fourier space. The assumption that f is real implies that $\overline{\hat{f}(p)} = \hat{f}(-p)$, where the bar denotes the complex conjugate. In particular, this relation shows that the Fourier space is covered twice if $\partial X^0 = \partial X$. Given the source-receiver configuration of a particular experiment, we can determine the domain in the Fourier space where the function $f^\wedge$ is known. This domain controls the spatial resolution of the reconstruction. In examples where the domain X is a half-space (given later in this section) we only have partial coverage since observation points are restricted to the boundary of the half-space. However, assuming infinite aperture we still obtain the function $f^\wedge$ over the whole Fourier space by continuing $f^\wedge$ with the help of the identity $\overline{f^\wedge}(p) = f^\wedge(-p)$.

We shall discuss the implications of our results in exploration geophysics and the comparison with existing migration schemes in greater detail elsewhere. Here we note only, that, in general, the construction of a GBO requires ray tracing and computation of solutions of the transport equation. However, in the case of constant background one can obtain analytical expressions for the GBO. Let us illustrate this with a few examples where an explicit construction of the GBO is available, and show that at least for these examples some of the migration schemes appearing in the literature are given by a generalized backprojection operator. We consider the case with a constant index of refraction and set

$$\hat{n} = \tilde{n} = 1,$$

without loss of generality.

Example 1: Let the domain X be the half-space $x_n \geqslant 0$ of the n-dimensional space $(x_1, x_2, \ldots, x_{n-1}, x_n)$ and suppose measurements are performed everywhere on the boundary $\partial X = \{(x_1, x_2, \ldots, x_{n-1}, 0)\}$ of the half-space X. Let $\xi = \eta = (\eta_1, \eta_2, \ldots, \eta_{n-1}, 0)$, so that we have coincident sources and receivers. The phase functions $\hat{\phi}$ and $\tilde{\phi}$ are

$$\hat{\phi} = \tilde{\phi} = |x - \eta|,$$

so that ϕ in (3.2) is

$$\phi(x,\eta) = 2|x - \eta|. \tag{6.5}$$

To compute the determinant h in (5.5) we make use of the identities

$$\phi_{x_i \eta_j} = -(4/\phi)\delta_{ij} - \phi_{x_i}\phi_{\eta_j}/\phi,$$

$$\phi_{x_n \eta_j} = -\phi_{x_n}\phi_{\eta_j}/\phi,$$

where δ_{ij} is the Kronecker symbol and $i,j = 1,2,\ldots,n-1$, and obtain

$$h(x,\eta) = 4^{n-1}\phi_{x_n}/\phi^{n-1}, \tag{6.6}$$

where

$$\phi_{x_n}(x,\eta) = 4x_n/\phi(x,\eta). \tag{6.7}$$

It follows from (2.4), (3.1), and (3.4) that

$$a(x,\eta,\eta) = 1/4(\pi\phi)^{n-1}, \tag{6.8}$$

where ϕ is given by (6.5). Using (6.6)–(6.8) the weight function b in (5.4) can be written as

$$b(x,\eta) = C_n(x_n/|x - \eta|),$$

where

$$C_n = 2^{2n+1}\pi^{n-1}.$$

We set

$$x_n/|x - \eta| = \bar{l}\cdot\bar{l}_0,$$

where $\bar{l}$ and $\bar{l}_0$ are unit vectors pointing in the direction of x_n axis and in the direction of the line connecting points x and η. The generalized backprojection operator R^* thus is

$$(R^*w)(x) = C_n \int_{\partial X} w(2|x - \eta|,\eta)\bar{l}\cdot\bar{l}_0\, d\eta, \tag{6.9}$$

where w is given in (6.2) or (6.3) depending on the dimension n. Here we integrate over all source-receiver positions on the boundary ∂X. The GBO in (6.9) can be considered as a migration scheme within our definition. This operator was obtained in Ref. 17 for the case n = 3 by a different approach.

Some of the migration schemes that have appeared in the literature also have a form of the GBO. Reference 3 is an example. However, the particular weight functions used are generally different from the one presented here.

Example 2: This example deals with the case where the source and receiver positions are confined to a sphere—the surface of the n-dimensional ball of the radius ρ. We can write $\xi = \eta = \rho\nu$, where ν is a unit vector indicating the position of the coincident source and receiver on the sphere. Thus, we have

$$\hat{\phi} = \tilde{\phi} = |x - \rho\nu|,$$

and

$$\phi(x,\nu) = 2|x - \rho\nu|. \tag{6.10}$$

The computation of the determinant in (5.5) yields

$$h(x,\nu) = \rho^{n-1}2^{2n}(\rho - x\cdot\nu)/\phi^n.$$

Since the weight function $a(x,\xi(\eta),\eta)$ in (5.2) can be written for this example as

$$a(x,\nu,\nu) = 1/4(\pi\phi)^{n-1},$$

we obtain the weight function

$$b(x,\nu) = C_n[\rho^{n-1}(\rho - x\cdot\nu)/|x - \rho\nu|],$$

and the GBO

$$(R^*w)(x) = C_n \int_{|\nu|=1} w(2|x - \rho\nu|,\nu) \times \frac{\rho^{n-1}(\rho - x\cdot\nu)}{|x - \rho\nu|}\, d\nu, \tag{6.11}$$

where w is given in (6.2) or (6.3) depending on the dimension n. The integration in (6.11) is over the unit sphere and $d\nu$ is the standard solid angle differential form. The GBO in (6.11) differs from the one constructed in Ref. 17 for the case n = 3. The approximation in (6.11) remains valid even if the point x is not close to the center of the ball.

Example 3: Finally, we consider the case where source and receiver positions are confined to the boundary of a half-space as in our first example. However, we assume now that sources and receivers are separated by a fixed distance $2d$.

Let η denote the coordinate of the midpoint between the source and receiver, so that we can write

$$\hat{\phi} = |x - \eta - d|, \tag{6.12}$$

$$\tilde{\phi} = |x - \eta + d|, \tag{6.13}$$

and

$$\phi(x,\eta) = |x - \eta - d| + |x - \eta + d|.$$

We consider the case of the dimension n = 2. Making use of the identities

$$\phi_{x_1} = \frac{x_1 - \eta - d}{\hat{\phi}} + \frac{x_1 - \eta + d}{\tilde{\phi}},$$

$$\phi_{x_2} = x_2\left(\frac{1}{\hat{\phi}} + \frac{1}{\tilde{\phi}}\right),$$

$$\phi_{x_1\eta} = -\left(\frac{1}{\hat{\phi}} + \frac{1}{\tilde{\phi}}\right) - \left(\frac{\hat{\phi}_{x_1}\hat{\phi}_\eta}{\hat{\phi}} + \frac{\tilde{\phi}_{x_1}\tilde{\phi}_\eta}{\tilde{\phi}}\right),$$

$$\phi_{x_2\eta} = -\left(\frac{\hat{\phi}_{x_2}\hat{\phi}_\eta}{\hat{\phi}} + \frac{\tilde{\phi}_{x_2}\tilde{\phi}_\eta}{\tilde{\phi}}\right),$$

we can, in this case, compute the determinant in (5.5) directly and obtain

$$h(x,\eta) = (\hat{\phi}_{x_2} + \tilde{\phi}_{x_2})\left(\frac{1}{\hat{\phi}} + \frac{1}{\tilde{\phi}}\right) + (\hat{\phi}_{x_2}\tilde{\phi}_{x_1} - \hat{\phi}_{x_1}\tilde{\phi}_{x_2})\left(\frac{\tilde{\phi}_\eta}{\tilde{\phi}} - \frac{\hat{\phi}_\eta}{\hat{\phi}}\right).$$

Substituting appropriate expressions for the derivatives of the phase functions we write h as

$$h(x,\eta) = \frac{x_2}{\hat{\phi}^2\tilde{\phi}^2}\left[(\hat{\phi} + \tilde{\phi})^2 - \frac{2d}{\hat{\phi}\tilde{\phi}}(d^2 + x_2^2 - 2(x-\eta)^2 d)\right].$$

Hence,

$$b(x,\eta) = \frac{8\pi x_2}{(\hat{\phi}\tilde{\phi})^{3/2}}\left[(\hat{\phi} + \tilde{\phi})^2 - \frac{2d}{\hat{\phi}\tilde{\phi}}(d^2 + x_2^2 - 2(x-\eta)^2 d)\right],$$

where $\hat{\phi}$ and $\tilde{\phi}$ are given in (6.12) and (6.13). The GBO in this case is

$$(R * w)(x) = \int_{-\infty}^{+\infty} w(2|x-\eta|,\eta) b(x,\eta) d\eta, \qquad (6.14)$$

where w is given in (6.3) for $n = 2$. It is easy to see that if $d = 0$ then we obtain the GBO in (6.9).

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APPENDIX A

The following lemma holds in the cases of Riemannian and Finsler spaces. It was used in Ref. 20 to prove a uniqueness theorem of the inverse travel time problem in the case of the Riemannian metric. We present here an elementary proof for the Euclidean space.

Lemma A: Let the function $\hat{\phi}$ satisfy the eikonal equation

$$(\nabla_x \hat{\phi}(x,\xi))^2 = n^2(x), \qquad (A1)$$

in the domain X with the boundary ∂X, where the parameter $\xi \in \partial X$. We assume that the boundary ∂X is diffeomorphic (with the preservation of orientation) to the unit sphere S_x^{n-1} centered at the origin of the tangent space at the point $x \in X$. (This unit sphere represents all directions at the point x.)

Let $\tilde{\phi}(x)$ have first partial derivatives and consider the determinant

$$J(x,\xi) = \begin{bmatrix} \tilde{\phi}_{x_1} & \tilde{\phi}_{x_2} & \cdots & \tilde{\phi}_{x_n} \\ \hat{\phi}_{x_1\xi_1} & \hat{\phi}_{x_2\xi_1} & \cdots & \hat{\phi}_{x_n\xi_1} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\phi}_{x_1\xi_{n-1}} & \hat{\phi}_{x_2\xi_{n-1}} & \cdots & \hat{\phi}_{x_n\xi_{n-1}} \end{bmatrix}, \qquad (A2)$$

where $\xi_1,\ldots,\xi_{n-1}$ is a local system of coordinates on the boundary. Then

$$J(x,\xi)d\xi_1 \cdots d\xi_{n-1} = n^{n-2}(\nabla_x\tilde{\phi}\cdot\nabla_x\hat{\phi})d\omega, \qquad (A3)$$

where $d\omega$ is the standard measure on the sphere S_x^{n-1}, and $\xi = \xi(\omega)$, where $\omega \in S_x^{n-1}$.

Proof: Since $\hat{\phi}$ satisfies the eikonal equation in (A1) we can write

$$\begin{cases} \hat{\phi}_{x_1} = n\cos\chi_1 \\ \hat{\phi}_{x_2} = n\sin\chi_1\cos\chi_2 \\ \vdots \\ \hat{\phi}_{x_{n-1}} = n\sin\chi_1\sin\chi_2\cdots\cos\chi_{n-1} \\ \hat{\phi}_{x_n} = n\sin\chi_1\sin\chi_2\cdots\sin\chi_{n-1}, \end{cases} \qquad (A4)$$

where $\chi_j = \chi_j(x,\xi_1,\ldots,\xi_{n-1})$, $j = 1,\ldots,n-1$, are angular coordinates on the unit sphere S_x^{n-1}. By substituting (A4) in the functional Jacobian in (A2) we obtain (A3). Let us carry out the calculation for the case $n = 3$. We have

$$\hat{\phi}_{x_1\xi_j} = -n\sin\chi_1\frac{\partial\chi_1}{\partial\xi_j},$$

$$\hat{\phi}_{x_2\xi_j} = n\cos\chi_1\cos\chi_2\frac{\partial\chi_1}{\partial\xi_j} - n\sin\chi_1\sin\chi_2\frac{\partial\chi_2}{\partial\xi_j}, \qquad (A5)$$

$$\hat{\phi}_{x_3\xi_j} = n\cos\chi_1\sin\chi_2\frac{\partial\chi_1}{\partial\xi_j} + n\sin\chi_1\cos\chi_2\frac{\partial\chi_2}{\partial\xi_j},$$

where $j = 1,2$.

Let us compute cofactors of the first row of the functional determinant. Using (A5) we compute the cofactor for the element $\tilde{\phi}_{x_1}$. We obtain

$$\begin{vmatrix} \hat{\phi}_{x_2\xi_1} & \hat{\phi}_{x_3\xi_1} \\ \hat{\phi}_{x_2\xi_2} & \hat{\phi}_{x_3\xi_2} \end{vmatrix} = n^2\left[\frac{\partial\chi_1}{\partial\xi_1}\frac{\partial\chi_2}{\partial\xi_2}\sin\chi_1\cos\chi_1 - \frac{\partial\chi_1}{\partial\xi_2}\frac{\partial\chi_2}{\partial\xi_1}\sin\chi_1\cos\chi_1\right] = n\frac{\partial(\chi_1,\chi_2)}{\partial(\xi_1,\xi_2)}\hat{\phi}_{x_1}\sin\chi_1.$$

Analogous calculations can be performed for all cofactors, so that we obtain (A3), where

$$d\omega = \sin\chi_1\, d\chi_1\, d\chi_2.$$

APPENDIX B

We briefly present here some first definitions and properties of pseudodifferential operators. Consider the operator $a(x,D)$,

$$(a(x,D)f)(x) = \int_{R^n} a(x,p) f\hat{\ }(p) e^{ip\cdot x}\, dp,$$

where $f\hat{}(p)$ denotes the Fourier transform of the function f. The function $a(x,p)$ is the symbol of the pseudodifferential operator $a(x,D)$.

Definition: Let Ω be an open subset of R^n and m be a real number. Let $S^m(\Omega)$ be the class of symbols and consist of infinitely differentiable functions $a(x,p)$, $a(x,p)\in C^\infty(\Omega \times R^n)$, such that to every compact $Q\subset\Omega$ and to every two multi-indices α, β there is a constant $C_Q(\alpha,\beta)$, such that

$$|\partial_p^\alpha \partial_x^\beta a(x,p)| \leqslant C_Q(\alpha,\beta)(1+|p|)^{m-|\alpha|}.$$

The pseudodifferential operator $a(x,D)$ is said to belong to the class $L^m(\Omega)$ if its symbol $a(x,p)$ belongs to $S^m(\Omega)$.

The following properties describe $a(x,D)$ as an operator.

If $a(x,p)\in S^m(\Omega)$ then $a(x,D)$ is a continuous operator

$$a(x,D): C_0^\infty(\Omega) \mapsto C^\infty(\Omega),$$

where $C_0^\infty(\Omega)$ denotes the class of infinitely differentiable functions with compact support in Ω. The operator $a(x,D)$ can be extended to a continuous map

$$a(x,D): \mathscr{E}'(\Omega) \mapsto \mathscr{D}'(\Omega),$$

where $\mathscr{D}'(\Omega)$ is the space of distributions on X [the dual of $C_0^\infty(\Omega)$] and $\mathscr{E}'(\Omega)$ is the space of distributions with compact support [the dual of $C^\infty(\Omega)$].

Definition: An operator is called to be regularizing if it maps

$$\mathscr{E}'(\Omega) \mapsto C^\infty(\Omega).$$

(This means that a regularizing operator transforms functions with singularities into infinitely smooth functions.)

Let $L^{-\infty}(\Omega)$ be the intersection of all $L^m(x)$, where m is real. One can prove that every operator from the class $L^{-\infty}(\Omega)$ is regularizing and every regularizing operator can be represented as an operator from the class $L^{-\infty}(\Omega)$.

The asymptotics in (4.19) and (5.14) have the following meaning in terms of classes of pseudodifferential operators: we can prove[14,16] that

$$T_j \in L^{-j}(\Omega),$$

for $j = 1,2,\ldots$, and

$$(F - I_{\partial X_\eta^0} - T_1 - T_2 - \cdots - T_l) \in L^{-l-1}(\Omega).$$

for $l = 0,1,2,\ldots$. In particular,

$$F - I_{\partial X_\eta^0} \in L^{-1}(\Omega).$$

This means that approximations in (4.20) and (5.16) allow reconstruction of discontinuities, since the discrepancy operator is a smoothing operator. It can be shown that an operator from the class $L^{-1}(\Omega)$ increases by 1 the number of derivatives of a function to which it is applied. In precise terms the following theorem holds.

Theorem: Let $a(x,D)$ be a pseudodifferential operator in Ω of the class $L^m(\Omega)$. Given any real number s the operator $a(x,D)$ can be extended as a continuous map

$$a(x,D): H^s_{\text{comp}}(\Omega) \mapsto H^{s-m}_{\text{loc}}(\Omega),$$

where $H^s_{\text{comp}}(\Omega)$ and $H^{s-m}_{\text{loc}}(\Omega)$ are the so-called Sobolev spaces of distributions.

The index s can be interpreted as a "number of derivatives." For detailed descriptions and proofs see Ref. 21 or any other reference on pseudodifferential operators.

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