i. [8 pts] Are there any points in the plate that are locally thicker or thinner than their nearby surroundings? If so, where are they? If not, explain why not.

ii. [5 pts] Is there a thinnest part of the plate? Do not find it, simply answer YES or NO and give a brief explanation justifying your answer.

SOLUTION:

(a) We need the gradient of the magnetic field.

$$\nabla B = \left\langle \frac{1}{xyz}(yz); \frac{1}{xyz}(xz); \frac{1}{xyz}(xy) \right\rangle = \left\langle \frac{1}{x}; \frac{1}{y}; \frac{1}{z} \right\rangle$$

The maximum rate of change of the magnetic field occurs in the direction of the gradient so the ship should be aimed in the direction

$$\nabla B(1;1;2) = \left\langle 1; 1; \frac{1}{2} \right\rangle = \mathbf{i} + \mathbf{j} + \frac{1}{2}\mathbf{k}$$

and the maximum rate of change of the magnetic field will be given by

$$\|\nabla B(1;1;2)\| = \sqrt{1^2 + 1^2 + \left(\frac{1}{2}\right)^2} = \frac{3}{2}$$

(b) i. We need to find and classify the critical points.

$$h_x = -2xe^y = 0 \implies x = 0$$

$$h_y = e^y\left(2y + y^2 - x^2\right) = 0 \implies 2y + y^2 = 0 \quad (\text{since } x = 0) \implies y = 0; -2$$

Critical points are $(0;0); (0;-2)$. Now apply the Second Derivatives Test.

$$D(0;0) = h_{xx}(0;0)h_{yy}(0;0) - [h_{xy}(0;0)]^2 = (-2)(2) - 0^2 = -4 < 0 \implies (0;0) \text{ is a saddle point}$$

$$D(0;-2) = h_{xx}(0;-2)h_{yy}(0;-2) - [h_{xy}(0;-2)]^2 = \left(-2e^{-2}\right)\left(-2e^{-2}\right) - 0^2 = 4e^{-4} > 0$$

$$\text{and } h_{xx}(0;-2) = -2e^{-2} < 0 \implies h(0;-2) \text{ is a local maximum}$$

The thickness is a local maximum at $(0;-2)$ so there is a point that is locally thicker than its nearby surroundings. There are no points in the plate that are locally thinner than their surroundings.

ii. YES. The thickness is a continuous function and the plate is a closed, bounded region so the Extreme Value Theorem applies. Since the interior critical points are a saddle and a local maximum, the thinnest part of the plate will be on the boundary.

3. [2350/031523 (25 pts)] Let $g(x;y) = \cos(\ xy\ ) +$

(b)

$$dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy = \quad y\sin(xy) + y^2 \; dx + [\quad x\sin(xy) + 2xy]\,dy$$

At $(1;1)$ we have $dg = 1\,dx + 2\,dy$ so that g is more sensitive to small changes in y.

(c) You arrive at the point $\frac{1}{2};1$ when $t = 4$ and the rate of change of temperature with respect to time is given by

$$\frac{dg}{dt} = \nabla g \;\; r'(t) = \langle \;\; y\sin(xy) + y^2; \quad x\sin(xy) + 2xy\rangle \qquad \frac{1}{2}t^{\;3/2}; \frac{t}{8}$$

$$\Rightarrow \quad \frac{dg}{dt}\Big|_{t=4} = \langle \quad (1)\sin(\;=2) + 1^2; \quad (1/2)\sin(\;=2) + 2(1/2)(1)\rangle \qquad \frac{1}{2}4^{\;3/2}; \frac{4}{8}$$

$$= \Big\langle \;1 \quad ; 1 \quad \frac{}{2}\Big\rangle \qquad \frac{1}{16}; \frac{1}{2} \quad = \quad \frac{1}{16} + \frac{}{16} + \frac{8}{16} \quad \frac{4}{16} = \frac{1}{16}(7 \quad 3\;)$$

Since this is negative, the temperature is decreasing at a rate of $\frac{1}{16}(7 \quad 3\;)$

4. [2350/031523 (16 pts)] A skateboard park has a new track in the form of the hyperbola $\frac{1}{4}x^2 \quad y^2 = 12$. You are standing at the point $(x;y) = (0;10)$

<u>SOLUTION:</u>

(a)

$$f\quad 1;\ \tfrac{1}{2}\ = 1$$

$$f_x(x;y) = \quad 2xe^{\ (x^2+2y)} \quad =) \quad f_x\ \ 1;\ \tfrac{1}{2} \ = \quad 2$$

$$f_{xx}(x;y) = \quad 2e^{\ (x^2+2y)} \quad 2x^2+1 \quad =) \quad f_{xx}\ \ 1;\ \tfrac{1}{2}\ = 2$$

$$f_{xy}(x;y) = 4\,xe^{\ (x^2+2y)} \quad =) \quad f_{xy}\ \ 1;\ \ ^{1}$$